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Journal of Geometry and Physics 27 (1998) 65–78

JOURNAL OF GEOMETRY AND PHYSICS

Properties of nonlinear Hodge fields

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Received 12 May 1997; received in revised form 22 August 1997

Abstract

Conditions are given for the global triviality and local smoothness of nonlinear Hodge fields having apparent singularities and, in some cases, nonuniform ellipticity. © 1998 Elsevier Science B.V.

Subj. Class.: Classical field theory
1991 MSC: 58E15, 81T13, 78A25
Keywords: Nonlinear Hodge equations; Gauge theory; Magnetism

1. Introduction

Define over an n-dimensional Riemannian manifold M a vector bundle X with compact structure group G. Let A be a connection 1-form on $\mathrm{ad}(X)$ with curvature 2-form B_A. In [11] we introduced the system

$$D_A B_A = D_A^*(\rho(Q)B_A) = 0, \tag{1.1}$$

where D_A^* is the formal adjoint of the exterior covariant derivative D_A on $\mathrm{ad}(X)$ and $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function satisfying

$$0 < \kappa_1 \le \rho^2(Q) + 2\rho(Q)\rho'(Q)Q \le \kappa_2 < \infty \tag{1.2}$$

for constants κ_1 and κ_2, where $Q = Q(B) = |B|^2$.

If G is abelian and $X = T^*M$, then Eqs. (1.1) reduce to the *nonlinear Hodge equations* [16]:

$$\mathrm{d}B = \delta(\rho(Q)B) = 0 \tag{1.3}$$

for the d-closed 2-form B, where d is the flat exterior derivative on M and δ its formal adjoint. If we interpret $\rho(Q)$ as $\mu^{-1}(Q)$, where μ denotes magnetic permeability, then

PII S0393-0440(97)00066-1

Eqs. (1.3) provide a model of magnetic fields in matter. We call solutions of (1.1) *nonlinear Hodge fields*, as Eqs. (1.1) can be obtained from Eqs. (1.3) by replacing the cotangent bundle with a curved bundle. If G is non-abelian but $\rho(Q)$ is constant, then Eqs. (1.1) reduce to the *Yang–Mills equations*: $DB = D^*B = 0$. If G is the abelian group $U(1)$ and $\rho(Q)$ is unity, then Eqs. (1.1) reduce to the *source-free Maxwell equations in free space*: $\mathrm{d}B = \delta B = 0$.

Eqs. (1.1) can be endowed with a variational structure by defining the *nonlinear Hodge energy*

$$E(B_A) = \int_M \left(\int_0^{Q(B_A)} \rho(s)\,\mathrm{d}s \right) \mathrm{d}M. \tag{1.4}$$

If G is abelian and $X = T^*M$, then the admissible class is a cohomology class of differential forms [18]. If G is non-abelian, typically $\mathrm{SO}(n)$, then the admissible class is a set of finite-energy connections (cf. [21] for the case $\rho \equiv 1$).

Eqs. (1.1) have a physical analogy as the Yang–Mills theory arising from the magnetostatics of materials possessing nonlinear dependence of magnetic permeability on applied magnetic field strength. The choice of non-abelian structure group puts a twist in the bundle of magnetization states. This twist is represented by the nonvanishing Lie bracket $[\ ,\ ]$ in the curvature 2-form $B_A = \mathrm{d}A + \frac{1}{2}[A, A]$. The existence in nature of precisely this sort of twisting is a matter for speculation.

In [11] we proved that if condition (1.2) is satisfied in a bounded open Lipschitz domain Ω of $\mathbb{R}^n$, then any weak solution (A, B_A) of (1.1) is Hölder continuous in Ω provided $B_A \in L^p(\Omega)$ for some $p > \frac{1}{2}n$. Moreover, if the quadratic form Q in (1.1) is replaced by the quantity $Q + m^2$, where m is a nonvanishing section of the $(1/n)th$ power of the determinant bundle, and if $\rho(Q) = (m^2 + |B_A|^2)^{\alpha-1}$ for $\alpha \in (\frac{1}{4}n, 1]$ where $n < 4$, then system (1.1) possesses a weak solution in Ω. Here we derive some fundamental global and local properties of solutions.

For details on the gauge-theoretic background of this paper see, e.g., [7]. The nonlinear Hodge equations (1.3) were introduced by Sibner and Sibner in [16]. For a brief description of the electromagnetic interpretation of the nonlinear Hodge equations see, e.g., [8]. Other applications of the nonlinear Hodge equations, and the decomposition theorem, were introduced in [18]. Throughout this paper we denote by C generic positive constants, the value of which may change from line to line.

2. A Liouville theorem for singular domains

The results proven in [11] were local. In this section we give an example of the simplest kind of global result available for solutions of (1.1). Let M be a complete Riemannian manifold having constant nonpositive sectional curvature. We place conditions on the integrability of the field B_A, and on the topology of its apparent singular set, sufficient to force the field potential A to be constant almost everywhere.

To this end we combine ideas from geometric measure theory due to Almgren [1] and Price [12] with p.d.e. methods of Carlson [22] and Serrin [13]. (See also [2].) We use geometric measure theory to replace the ordinary derivative of the energy functional E by a suitably defined Lie derivative, so that the variations of E occur in a space of reparametrizations of the underlying manifold rather than in a space of infinitesimal deformations of the bundle connection. The former space is more tolerant of singularities than is the latter space. The p.d.e. methods are used to relate a priori information on the integrability of B_A to topological information about the singular set. If B_A is sufficiently integrable, then the existence of an apparent singularity is no longer an obstruction to performing a limiting argument that implies global triviality of the field. The results of this section extend earlier work for mappings between Riemannian manifolds [10].

The hypotheses which we place on the field B_A are extremely weak (but so is the conclusion). For example, we do not expect B_A to satisfy a differential equation, even weakly. Instead we place a condition on the energy – expression (2.1), below – analogous to prescribing the decay of a generalized current density (cf. [12]) or of a generalized mean curvature (cf. [1, Section 4.3]). This analogy between current density and mean curvature will be developed briefly in Section 5.

Denote by ϕ^t a 1-parameter family of compactly supported diffeomorphisms of M with $\phi^s \circ \phi^t = \phi^{s+t}$ and ϕ^0 the identity transformation. The family ϕ^t can be lifted to the principal bundle Π by parallel transport, with respect to an arbitrary smooth connection, along the curve $x_s = \phi^s(x)$ from $x_0 = x$ to $x_t = \phi^t(x)$. For $A \in \Gamma(M, \Lambda^1(\mathrm{ad}\,\Pi))$, define $A^t = (\psi^t)^* A$, where ψ^* is a lifting of ϕ^* to $\Gamma(M, \Lambda^1(\mathrm{ad}\,\Pi))$. Details of this construction are given in [12]. The *r-variation* of the nonlinear Hodge energy E is given by the expression

$$\delta_r E(B_A) = \frac{\mathrm{d}}{\mathrm{d}t} E(B_{A^t})_{|t=0},$$

where E is the nonlinear Hodge energy (1.4). We say that A is an *r-stationary connection* if $\delta_r E = 0$. In proving our Liouville theorem we only require the weaker assumption that on any domain $\Omega \subset M/\Sigma$ for which $\partial\Omega = 0$ we have

$$\delta_r E(B_A)_{|\Omega} \geq -\int_\Omega f|\xi| \int_0^{Q(B_A)} \rho(s)\,\mathrm{d}s\,\mathrm{d}M, \tag{2.1}$$

where f is a measurable function, with compact support in a disc of radius τ, constructed so that $f(x, \tau) = \mathrm{o}(1)$ as $\tau \to \infty$ (this limit exists, as M is H^n or $\mathbb{R}^n$); the *variation vector field* $\xi = \mathrm{d}\phi(x, t)/\mathrm{d}t_{|t=0}$ is the initial velocity field for the flow generated by ϕ^t. As in [11], we assume that condition (1.2) is satisfied by the function ρ. The constants κ_1 and κ_2 in this condition are important, for we do not assume that the energy E is finite on all of M, but rather that the energy satisfies a growth condition at infinity. In this section we prove:

Theorem 2.1. *Let the pair* (A, B_A) *satisfy* (1.2), (2.1), *and the growth condition*

$$E(B_A)_{|D_\tau} = \mathrm{o}[\tau^{n-(4\sqrt{\kappa_2/\kappa_1})}] \tag{2.2}$$

as τ tends to infinity, where D_τ is an n-disc of radius τ contained in an n-dimensional euclidean or hyperbolic space denoted by M (topologically $\mathbb{R}^n$ with constant nonpositive sectional curvature); let A have a possible singularity on a compact subset Σ of codimension $k\epsilon(2, n]$; let $n \geq 4(\kappa_2/\kappa_1)^{1/2}$, where κ_1 and κ_2 are the constants of inequality (1.2); define f to be $\kappa_3/|x|\tau$ if $0 < |x| \leq \tau$ and 0 if $|x| > \tau + \epsilon$ for some $\epsilon > 0$. If $B_A \epsilon L^p(M)$ for $p = 2(k - \epsilon_0)/(k - 1 - \epsilon_0)$, $\epsilon_0 \in (0, k - 1)$, then B_A vanishes almost everywhere on M.

If Σ is a Lipschitz submanifold of M then ϵ_0 can be taken to be zero. Notice we do *not* assume that the field B_A satisfies Eqs. (1.1) anywhere in M. In Section 5 we consider an extension in which assumption (1.2) of Theorem 2.1 can be weakened. For simplicity, and with no essential reduction of generality, we take M to be $\mathbb{R}^n$ in the proof; for the modifications necessary in order to include hyperbolic space, see [12]. Because f is only o(1), condition (2.2) implies that E may not be stationary even as τ tends to infinity.

Lemma 2.2. *If a gauge potential A has finite energy in a compactly supported domain Ω of $\mathbb{R}^n$, then*

$$\delta_r E = -\int_\Omega \left(\int_0^{Q(B)} \rho(s)\,\mathrm{d}s \right) \operatorname{div}\xi * 1 + 4\int_\Omega \rho(Q)\langle B(\nabla_i\xi, e_j), B(e_i, e_j)\rangle * 1,$$

where $\{e_i\}$, $i = 1, \ldots, n$, is an orthonormal basis for the tangent space of Ω and ∇_i is the derivative in the direction of the vector e_i.

Proof. Write $u \equiv \phi^t(x) = x + t\xi(x) + \mathrm{O}(t^2)$. Recognizing that terms of second-order and higher in t vanish, we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t|_{t=0}} B_{ij}(u)\,\mathrm{d}u^i\,\mathrm{d}u^j &= \frac{\mathrm{d}}{\mathrm{d}t|_{t=0}} B_{ij}(u)\frac{\partial u^i}{\partial x^k}\,\mathrm{d}x^k\,\frac{\partial u^j}{\partial x^m}\,\mathrm{d}x^m \\ &= \frac{\mathrm{d}}{\mathrm{d}t|_{t=0}} B_{ij}(u)\left(\delta^i_k\delta^j_m + \delta^i_k t\frac{\partial\xi^j}{\partial x^m} + \delta^j_m t\frac{\partial\xi^i}{\partial x^k}\right)\mathrm{d}x^k\,\mathrm{d}x^m.\end{aligned}$$

Employing obvious symmetries, we have

$$\frac{\mathrm{d}}{\mathrm{d}t|_{t=0}} B_{ij}(u)\,\mathrm{d}u^i\,\mathrm{d}u^j = 2B_{ij}(x)\frac{\partial\xi^i}{\partial x^k}\,\mathrm{d}x^k\,\mathrm{d}x^j.$$

Observe that

$$\delta_r E(B_A)_{|\Omega} = \frac{\mathrm{d}}{\mathrm{d}t|_{t=0}}\int_\Omega R(\langle\Phi^{t^*}B_A, \Phi^{t^*}B_A\rangle) * 1,$$

where

$$R(Q) \equiv \int_0^Q \rho(s)\,\mathrm{d}s.$$

The chain rule implies that if $J = |\partial x/\partial u|$ is the Jacobian, then

$$\delta_r E(B_A)_{|\Omega} = \int\limits_{\Omega} R(Q) \frac{\mathrm{d}}{\mathrm{d}t|_{t=0}} J[(\phi^t)^{-1}] * 1 + \int\limits_{\Omega} R'(Q) 2\left\langle \frac{\mathrm{d}}{\mathrm{d}t|_{t=0}} [B_{ij}(u)\, \mathrm{d}u^i\, \mathrm{d}u^j],\, B_{lm}(u)\, \mathrm{d}u^l\, \mathrm{d}u^m \right\rangle * 1,$$

from which the assertion follows. □

Lemma 2.3. *Under the hypotheses of Theorem 2.1, the nonlinear Hodge energy satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\tau^{4\sqrt{\kappa_2/\kappa_1}-n} \mathrm{e}^{-\kappa_3/\tau} \int\limits_{B_\tau} R(Q) * 1\right] \geq 0,$$

where D_τ is an n-disc of radius τ centered at the origin of coordinates in $\mathbb{R}^n$.

Proof. We initially prove the lemma under the assumption that (A, B_A) satisfies (1.2), (2.1), and (2.2) on all of $D_{\tau+\delta}$, where δ is a sufficiently small positive constant. We then introduce modifications necessary to accommodate the possible existence of the set Σ as a subset of D_τ. Thus for the moment consider the special case in which A is nonsingular on $D_{\tau+\delta}$.

Choose $\xi = \eta(r)r \cdot (\partial/\partial r)$, where $\{e_i\}_{i=1}^n = \{\partial/\partial_r, \partial/\partial\theta_2, \ldots, \partial/\partial\theta_n\}$ is an orthonormal basis; $\eta(r) \in C_0^\infty[0, 1]$ is a function chosen so that $\eta(r) = v(r/\tau)$ satisfies $\eta'(r) \leq 0\ \forall r$; $\eta(r) = 1$ for $r \leq \tau$, $\tau \in (0, 1)$; and $\eta(r) = 0$ for $r > \tau + \delta$, $\delta > 0$. In terms of this choice of variation vector field,

$$\delta_r E = -\int\limits_{\Omega} R(Q)[n\eta + r\eta'] * 1 + 4\int\limits_{\Omega} \rho(Q)\left[\eta|B|^2 + r\eta'\left|\frac{\partial}{\partial r}\lrcorner B\right|^2\right] * 1.$$

Thus (2.1) can be written

$$\int\limits_{\Omega} R(Q)[n\eta + r\eta' - fr\eta] * 1 \leq 4\int\limits_{\Omega} \rho(Q)\left(Q\eta + r\eta'\left|\frac{\partial}{\partial r}\lrcorner B\right|^2\right) * 1. \tag{2.3}$$

Write condition (1.2) in the form of a differential inequality

$$0 < \kappa_1 \leq \frac{\mathrm{d}}{\mathrm{d}s}[s\rho^2(s)] \leq \kappa_2 < \infty.$$

Integrating this expression over s from 0 to Q yields $\kappa_1 Q \leq Q\rho^2(Q) \leq \kappa_2 Q$. For positive values of Q and, by a limiting argument, for nonnegative values of Q, we immediately obtain $\rho(Q) \leq \sqrt{\kappa_2}$. Alternatively, we can integrate the differential form of (1.2) over s from 0 to t to establish the lower bound $\rho(t) \geq \sqrt{\kappa_1}$, which can itself be integrated over t

from 0 to Q to obtain $R(Q) \geq \sqrt{\kappa_1} Q$. We use the latter inequality to bound the right-hand side of (2.3) above by the expression

$$4 \int_\Omega \rho(Q) \frac{R(Q)}{\sqrt{\kappa_1}} \eta * 1 + 4 \int_\Omega \rho(Q) r \eta' \left| \frac{\partial}{\partial r} \lrcorner B \right|^2 * 1$$

and the former inequality to bound *this* expression above by

$$4\sqrt{\frac{\kappa_2}{\kappa_1}} \int_\Omega R(Q) \eta * 1 + 4 \int_\Omega \rho(Q) r \eta' \left| \frac{\partial}{\partial r} \lrcorner B \right|^2 * 1.$$

Inequality (2.3) now assumes the form

$$\int_\Omega R(Q)[(n - 4\sqrt{\kappa_2/\kappa_1})\eta + r\eta' - fr\eta] * 1 \leq 4 \int_\Omega \rho(Q) r \eta' \left| \frac{\partial}{\partial r} \lrcorner B \right|^2 * 1.$$

Writing $r\eta'(r) = -\tau(\partial/\partial\tau)\eta_\tau(r)$ and multiplying both sides of the resulting inequality by the integrating factor

$$\tau^{4\sqrt{\kappa_2/\kappa_1} - (n+1)} \mathrm{e}^{-\kappa_3/\tau}$$

completes the proof for the special case of a nonsingular gauge potential.

In the general case A is singular on $\Sigma \subset D_\tau$ and $B_A \in L^p(M)$ for $p = 2(k - \epsilon_0)/(k - \epsilon_0 - 1)$. (The requirement that Σ is contained in D_τ involves no loss of generality, as the same arguments will work, with slightly increased notational complexity, if only part of Σ is contained in D_τ.)

Replace the variation field of the preceding case by the quantity $\xi = \chi^{(\nu)}\eta(r)r \cdot (\partial/\partial r)$. We have included in ξ an extra multiplicative term $\chi^{(\nu)}$, representing a sequence of functions which vanish in the neighborhood of Σ. By a theorem of Serrin [13, Lemma 8], if the compact subset Σ has zero s-capacity with respect to M, where $1 \leq s \leq n$, then we can choose the sequence of functions $\chi^{(\nu)}$ in such a way that $0 \leq \chi^{(\nu)} \leq 1 \; \forall \nu$; $\lim_{\nu\to\infty} \chi^{(\nu)} = 1$ a.e.; and $\lim_{\nu\to\infty} \nabla\chi^{(\nu)} = 0$ in L^s. Moreover, a result by Carlson [22] guarantees that if the compact subset Σ has Hausdorff dimension m, for $0 < m < n - 1$, then Σ thas zero s-capacity with respect to the n-dimensional set M, where $s = n - m - \epsilon_0$ and ϵ_0 is a number in $(0, n - m - 1)$.

Our choice of variation vector field results in a harmless multiplicative term $\chi^{(\nu)}$ which must be carried throughout the calculation, and a possibly dangerous term involving the derivative of $\chi^{(\nu)}$ with respect to r, which enters into the derivatives of ξ via the product rule. Thus both sides of (2.3) now contain an extra term which must be estimated. The left-hand side of (2.3) is replaced by the quantity

$$\int_\Omega R(Q)[\chi^{(\nu)}(r)(n\eta + r\eta') + \chi^{(\nu)'}(r) r\eta - fr\eta] * 1.$$

Writing $\rho(t) \leq \sqrt{\kappa_2}$ and integrating over t from 0 to Q yields

$$\int_{\Omega} \chi^{(\nu)'}(r) r\eta R(Q) * 1 \geq -\sqrt{\kappa_2}\|\nabla\chi^{(\nu)}\|_{L^p}\|Q\|_{L^q}$$

for $p = k - \epsilon_0$ and $q = (k - \epsilon_0)/(k - \epsilon_0 - 1)$. The right-hand side of this inequality tends to zero in the limit as ν tends to infinity. Thus the left-hand side of (2.3) only becomes stronger if we neglect the extra term.

Similarly, on the right-hand side in (2.3) we now have

$$\int_{\Omega} \rho(Q)\chi^{(\nu)'}(r) r\eta \left|\frac{\partial}{\partial r} \rfloor B\right|^2 * 1 \leq \sqrt{\kappa_2}\|\nabla\chi^{(\nu)}\|_{L^p}\|Q\|_{L^q},$$

which tends to zero as ν tends to infinity for the same values of p and q used in estimating the left-hand side of (2.3). We complete the proof as in the preceding case, by the method of integrating factors. This completes the proof of Lemma 2.3. □

Combining the assertion of Lemma 2.3 with the energy condition (2.2) completes the proof of Theorem 2.1.

3. Removable singularities in nonlinear Hodge forms

Henneaux and Teitelboim [4] have developed a model for electrodynamics in which the gauge potential is a differential form of arbitrary degree. The gauge group is $U(1)$. The classical magnetostatics of this theory for magnetic media are mathematically identical to nonlinear Hodge theory. In this section we take the B-field to be a differential form of degree w on a Riemannian manifold M. Isolated point singularities in this field would correspond physically to "magnetic charges," or "sources of true magnetism." We show that apparent singularities of this kind are removable under certain hypotheses on the integrability of the field, the size of the singularity, and the dimension of the underlying domain. In this section we prove:

Theorem 3.1. *Let the differential form $B \in \Gamma(M, \Lambda^w(T^*M))$ smoothly satisfy Eqs.* (1.3) *with the ellipticity condition* (1.2) *on a disc D/Σ. Here D is a small euclidean n-disc, $n > 2(w+1)$, centered at the origin of coordinates of a coordinate chart S on M and completely contained in S; Σ is a compact set of codimension $k > 2n/(n-2w)$ completely contained in D. If $B \in L^{n/w}(D)$, then $B \in C^{0,\gamma}(D)$ for some $\gamma > 0$.*

Lemma 3.2. *Let $B \in \Gamma(M, \Lambda^w(T^*M))$ smoothly satisfy the inequality*

$$\int_D \left\{ \sum_{i,j=1}^n a^{ij} \frac{\partial Q}{\partial x^i}\frac{\partial \zeta}{\partial x^j} + \zeta \left(\sum_{j=1}^n b^j \frac{\partial Q}{\partial x^j} - zQ \right) \right\} * 1 \leq 0 \tag{3.1}$$

on a disc D/Σ, with a^{ij} satisfying the ellipticity condition $m_1|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq m_2|\xi|^2$, where m_1 and m_2 are positive constants; D is a small euclidean n-disc, $n > 2(w+1)$, of *radius τ, centered at the origin of coordinate of a coordinate chart S on M and completely contained in S; Σ is a compact set, completely contained in D, of codimension k, where $n \geq k > 2n/(n-2w)$. Assume that z lies in the space $L^{n/2}(D)$ and that the functions b^j are in $L^p(D)$ for some $p > n$. If $B \in L^{n/w}(D)$, then B is bounded in all of D.*

Proof. The proof is an extension of arguments in [3,13,15]. Choose $\zeta = (\eta\psi)^2 G(Q)$, where $\eta, \psi \geq 0$ but $\psi(x) = 0\ \forall x \in \Omega(\Sigma)$, where Ω is a neighborhood of Σ; $\eta \in C_0^\infty(D')$ for $D' \subset\subset D$; $G(Q) = H(Q)H'(Q)$, where $H(Q)$ is the following variant of Serrin's test function:

$$H(Q) = \begin{cases} Q^{[n/(n-2)]^i n/4w} & \text{for } 0 \leq Q \leq l, \\ \dfrac{k-\epsilon}{k-2-\epsilon}\left[(l \cdot Q^{(k-2-\epsilon)/2})^{[n/(n-2)]^i n/2w(k-\epsilon)} - \dfrac{2}{k-\epsilon} l^{[n/(n-2)]^i n/4w}\right] \\ \text{for } l \leq Q. \end{cases}$$

Iterate a sequence of elliptic estimates, taking successively $B \in L^{\alpha(i)}(D)$ for $\alpha(i) = [n/(n-2)]^i (n/w)$, $i = 0, 1, 2, \ldots$ Letting $\|\cdot\|_{p,q}$ denote $H^{p,q}$ norm we have, by ellipticity,

$$\|(aG')^{1/2}\eta\psi\nabla Q\|_{0,2} \geq C\|\eta\psi\nabla H\|_{0,2},$$

$$\|a\nabla Q \cdot 2\eta\psi\nabla(\eta\psi)HH'\|_{0,1} \leq \epsilon\|\eta\psi\nabla H\|_{0,2}^2 + C(\epsilon)\|\nabla(\eta\psi)H\|_{0,2}^2,$$

$$\|zQ(\eta\psi)^2 G\|_{0,1} \leq C\|z\|_{0,n/2}\|\eta\psi H\|_{1,2}^2,$$

$$\|b\nabla Q(\eta\psi)^2 HH'\|_{0,1} \leq C(\epsilon)\|b\eta\psi H\|_{0,2}^2 + \epsilon\|\eta\psi\nabla H\|_{0,2}^2.$$

Replace ψ by a sequence ψ_j of functions, vanishing on Σ, such that as $j \to \infty$ the sequence $\psi_j \to 1$ in L^∞ but $\nabla\psi_j \to 0$ in $L^{k-\epsilon}$. We find that for every l we have

$$\begin{aligned} 0 &\leq \lim_{j\to\infty} \|\eta(\nabla\psi_j)H\|_{0,2}^2 \\ &\leq \lim_{j\to\infty} C(l)\|\nabla\psi_j\|_{0,k-\epsilon}^2 \|B\|_{0,\alpha(i)}^{\alpha(i)(k-2-\epsilon)/(k-\epsilon)} \leq 0. \end{aligned}$$

Substituting these estimates into (3.1), absorbing small terms on the left, and letting l tend to infinity, we obtain on some smaller disc the estimate $\|\eta\nabla Q^{h(i)}\|_{0,2} \leq C\|(\nabla\eta)Q^{h(i)}\|_{0,2}$ for $h(i) = \frac{1}{4}\alpha(i)$. The Sobolev embedding theorem allows us to iterate this sequence of inequalities with the value of i increased by 1 at each iteration. We conclude after a finite number of iterations that $Q^q \in H^{1,2}(D_{r\tau})$ for any finite positive q and some $r \in (0, 1]$. Because B is smooth away from the singularity and Σ is compact, we find that $Q^q \in H^{1,2}$ on all of D. This is sufficient to apply Theorem 5.3.1 of Morrey [9] with $\lambda = h(i_N)$, where N depends on n, proving Lemma 3.2. □

Proof of Theorem 3.1. We observe that solutions of Eqs. (1.3) locally satisfy inequality (3.1) with considerably stronger hypotheses on a^{ij}, b^j and z than appear in Lemma 3.2 (cf. [14;

17, Sections 3, 4; 19, Section 1] for details; we assume these relatively weak hypotheses in Lemma 3.2 not only for mathematical generality, but also to derive a subsequent gauge improvement theorem for solutions of (1.1) possessing higher-order singularities (Theorem 4.2)). Moreover, the result of Lemma 3.2 implies, via integration by parts, that B is a weak solution of Eqs. (1.3) in all of D. Now apply Theorem 4.1 of [14] to conclude that B is Hölder continuous on D. This concludes the proof of Theorem 3.1. □

4. Gauge improvement in nonlinear Hodge fields

One criterion for selecting the hypotheses of the existence and regularity theorems in [11] was that we do not expect better results for system (1.1) than we have for the standard Yang–Mills or nonlinear Hodge equations. For example, if we let κ_1 tend to zero in condition (1.2), then a regularity theorem would apparently require the solution of boundary-value problems for the higher-dimensional Yang–Mills equations and the nonlinear Hodge equations for 2-forms (in order to apply the standard regularization argument [14]). Both are well-known open problems. Nor can we weaken the L^p hypothesis on B_A $(p > \frac{1}{2}n)$ without improving the existing regularity theory for Yang–Mills connections [21]. However, we can weaken the L^p hypotheses if we assume that the solution (A, B_A) is actually smooth on some subdomain. We can produce solutions of the Yang–Mills equations which are smooth on a subdomain by taking the limit of a sequence of approximations, and solutions of (1.1) can also be obtained in this way. We expect that the sequences will converge only on subsets of the original domain. This process will result in solutions of (1.1) which possess singularities. By standard arguments we can predict the Hausdorff dimension of the singular set, which will depend on the choice of ρ. The question is whether apparent singularities can be removed by applying a continuous gauge transformation.

For example, Theorem 3.1 can be extended to solutions of Eqs. (1.1). In place of inequality (3.1) we use the following estimate (cf. [11, Section 3]):

Proposition 4.1. *Solutions (A, B_A) of Eqs. (1.1) and condition (1.2) on a domain $\Omega \subset \mathbb{R}^n$ satisfy the differential inequality*

$$\sum_{i,j=1}^{n} -\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial Q}{\partial x^i}\right) - C_1(|A| + |\theta|)\sum_{j=1}^{n}\left|\frac{\partial Q}{\partial x^j}\right| - C_2(|\nabla A| + |B_A|)Q \le 0.$$

Here C_1 and C_2 are positive dimensional constants, θ is a section of the determinant bundle raised to the power $1/n$, and $a^{ij}(B)$ are bounded measurable functions satisfying the ellipticity condition of Lemma 3.2.

Proof. The Laplace–Beltrami operator ∇^2 satisfies

$$\begin{aligned}-\tfrac{1}{2}\nabla^2\langle\rho(Q)B_A, \rho(Q)B_A\rangle &= -\langle\nabla^2(\rho(Q)B_A), \rho(Q)B_A\rangle \\ &\quad + |\nabla(\rho(Q)B_A)|^2 + \rho^2(Q)K(Q).\end{aligned}$$

In, e.g., [6], $K(\cdot)$ is explicity computed. We require only the observation that K is a quadratic form in B_A. Now B_A satisfies (1.1), so we have

$$-\tfrac{1}{2}\nabla^2\langle\rho(Q)B_A,\rho(Q)B_A\rangle = \langle-\delta d(\rho(Q)B_A),\rho(Q)B_A\rangle + \langle d*[A,*(\rho(Q)B_A)],\rho(Q)B_A\rangle + |\nabla(\rho(Q)B_A)|^2 + \rho^2(Q)K(Q),$$

where $*: \Lambda^p \to \Lambda^{n-p}$ is the Hodge involution. We estimate

$$\langle d*[A,*(\rho(Q)B_A)],\rho(Q)B_A\rangle \le \kappa_2(|A||\nabla Q| + |\nabla A|Q)$$

from the product rule and the inequality

$$\langle(\nabla\rho(Q)B_A)\rangle,\rho(Q)B_A\rangle \le |\rho'(Q)\cdot(\nabla Q)Q\rho(Q) + \tfrac{1}{2}\rho^2(Q)\cdot\nabla Q| \le \tfrac{1}{2}(\rho^2(Q) + 2\rho(Q)\rho'(Q)Q)\nabla Q| \le \tfrac{1}{2}\kappa_2|\nabla Q|.$$

There exist bounded functions b^j, $j = 1,\dots,n$, such that

$$-\tfrac{1}{2}\nabla^2(\rho^2 Q) = -\tfrac{1}{2}(\rho^2 + 2\rho\rho' Q)\nabla^2 Q + \sum_{j=1}^{n} b^j(x^1,\dots,x^n)\frac{\partial Q}{\partial x^j}.$$

Taken together, these statements verify the assertion of Proposition 4.1. □

Theorem 4.2. *If the pair (A, B_A) is a smooth solution of (1.1) in D/Σ with ρ satisfying condition (1.2), if A is an element of the space $H^{1,n/2}(D)$, if B is an element of $L^{n/2}(D)$, and if D, Σ, and n satisfy the hypotheses of Theorem 3.1 with $w = 2$ and Σ a Lipschitz manifold, then there exists a continuous gauge transformation g to a gauge in which the pair $(g(A), B)$ is Hölder continuous in all of D.*

Remark. The condition on A requires fixing a gauge in which A is already reasonably smooth. Theorem 4.2 then guarantees the accessibility of a gauge in which A is considerably smoother. For this reason we call such a result a *gauge improvement theorem* rather than a removable singularities theorem. Unlike Yang–Mills fields, solutions of (1.1) fail to satisfy an elliptic system with diagonal principal part, even in a good gauge. (The same is true of solutions of the nonlinear Hodge equations.) Thus, for example, Hölder continuity does not automatically imply any higher degree of smoothness.

Proof of Theorem 4.2. Apply the arguments of Lemma 3.2 with inequality (3.1) replaced by the inequality of Proposition 4.1. Choose $w = 2$, $|b| = |A| + |\theta|$, and $z = |\nabla A| + |B|$. We find that $|B|^{n/4}$ lies in the space $H^{1,2}(D)$. But this implies, via the Sobolev embedding theorem, that B lies in a higher L^p space than $p = \frac{1}{2}n$. An elementary integration by parts against an admissible test function shows that B is a weak solution of (1.1) in all of D. Now apply the arguments of [11] for weak solutions which lie in $L^p(D)$ for $p > \frac{1}{2}n$. Use the modifications of [21] introduced in [20] in order to make gauge transformations up to the boundary of Σ. (This is where we use the fact that Σ is a Lipschitz manifold.) This completes the proof. □

5. A nonuniformly elliptic example

In this section we develop a formal analogy between a simple model for magnetic fields in permanent magnets and certain hypersurfaces of prescribed mean curvature. Consider the equations for magnetic media where $\rho(Q) = \mu^{-1}(Q)$. Here μ is the magnetic permeability of the medium, assumed to depend on the quadratic form $Q = |B|^2$, where B is the magnetic field (see, e.g., [5, p. 251]). If the medium possesses nonzero current density, represented in the Henneaux–Teitelboim model by a w-form J, then the equations become

$$\mathrm{d}B_A = 0, \qquad \delta(\mu^{-1}B_A) = J. \tag{5.1}$$

In the special case in which $w = 0$ (i.e., A and J are 0-forms and B is a 1-form) and $\mu(Q) = (Q+1)^{1/2}$, Eqs. (5.1) are identical to the equations for n-dimensional hypersurfaces of prescribed mean curvature J [18]. Because the pre-saturation permeability of a magnetic material always exceeds its vacuum value of unity and tends to increase with increasing B at a decreasing rate, this choice corresponds to one of the simplest physical models for the dependence of μ on Q.

Precisely, let Ω be a bounded, simply connected, convex domain of $\mathbb{R}^n$ having prescribed boundary data $\phi : \partial\Omega \to \mathbb{R}^{n-1} \in C^{2+\gamma}(\partial\Omega)$ for some $\gamma > 0$. Let $\Sigma : x^{n+1} = A(x^1, \ldots, x^n)$ be a family of n-dimensional hypersurfaces defined by the vector-valued function A and passing through the family of $(n-1)$-dimensional hypersurfaces defined by the vector-valued function ϕ. In this case Eqs. (5.1) define a family A of hypersurfaces having gradients B and mean curvatures equal to the prescribed vector-valued function $J \in C^1(\Omega)$. Eqs. (5.1) are elliptic provided that

$$0 < \rho(Q) + 2Q\rho'(Q) = \mu^{-1}(Q)\left(1 - \frac{2Q}{\mu(Q)}\frac{\mathrm{d}\mu}{\mathrm{d}Q}\right). \tag{5.2}$$

Notice that if $\mu(Q) = C(Q+1)^{1/2}$ for an arbitrary nonzero constant C, then the right-hand side of Eq. (5.2) is equal to $C'(Q+1)^{-3/2}$ for some positive constant C'. This expression exceeds zero for $Q < \infty$. However, ellipticity breaks down in the limit as Q tends to infinity.

The breakdown of ellipticity is a serious problem for arguments such as those of Section 3 and we do not expect good results in such cases unless the conditions listed in the appendix to [14] are met. However, one might expect that the arguments of Section 2 would be immune to a breakdown of ellipticity in the variational equations, as Theorem 2.1 does not require the field to satisfy any differential equation at all.

In fact, the ellipticity condition (1.2) is a more important hypothesis for Theorem 2.1 than the satisfaction of the variational equation that the condition qualifies. A breakdown of ellipticity implies that the constant κ_1 in condition (1.2) equals zero. In this case the left-hand side of the inequality of Lemma 2.3, which is the foundation of the proof of Theorem 2.1, fails to exist as a mathematical expression.

Nevertheless, a slightly amended version of Theorem 2.1 can be established for dimensions exceeding 4 in which condition (1.2) is replaced by the condition $\lim_{Q\to\infty}[\rho(Q) + 2Q\rho'(Q)] = 0$ in the special case

$$\rho(Q) = \mu^{-1}(Q) \propto (Q+1)^{-1/2}. \tag{5.3}$$

In this section we prove:

Theorem 5.1. *Assume the hypotheses of Theorem* 2.1 *for the w-form B, $w \in \mathbb{N}$, except for conditions* (1.2), (2.2), *and the hypothesis on dimension. Replace* (2.2) *by the condition $E(B_A)_{|D} = o[\tau^{n-2w}]$ as τ tends to infinity, where D is an n-disc of radius τ and $n > 2w$. In addition, let $\rho(Q)$ satisfy* (5.3) *with positive proportionality constant. Then B vanishes almost everywhere on M.*

Proof. We have $Q(1+Q)^{-1/2} \le 2[(1+Q)^{1/2}-1]$, implying via (5.3) that $Q\rho(Q) \le R(Q)$. Then

$$2w\int_{\Omega}\rho(Q)\left(Q\eta + r\eta'\left|\frac{\partial}{\partial r}\lrcorner B\right|^2\right) * 1 \le 2w\int_{\Omega} R(Q)\eta + \rho(Q)r\eta'\left|\frac{\partial}{\partial r}\lrcorner B\right|^2 * 1.$$

Substituting this result into (2.3), we continue as in the uniformly elliptic case but with a different integrating factor, to obtain in place of Lemma 2.3 the inequality

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tau^{2w-n}\mathrm{e}^{-\kappa_3/\tau}\int_{D\tau}\int_0^{Q(B)}\rho(s)\,\mathrm{d}s * 1\right) \ge 0.$$

This completes the proof of Theorem 5.1. □

If B is a 1-form, Theorem 5.1 applies to hypersurfaces of prescribed mean curvature; the variational inequality (2.1), with the decay condition on f, is in this case a condition on the decay at infinity of the mean curvature f of the hypersurface. If B is a 2-form, the theorem applies to our generalized model of magnetostatics for a special (simple) choice of $\mu(Q)$. The variational inequality (2.1), with the decay condition on f, is in this case a condition on the decay at infinity of generalized current density f. Obviously, the zero-current case in the magnetostatics model corresponds to the minimal surface case in the prescribed mean curvature model.

Appendix A

We take this opportunity to provide a list of errata for the first article in this sequence: *Yang–Mills fields with nonquadratic energy*, J. Geom. Phys. 19 (1996) 379–398. None of the corrections given here affects the truth of any of the theorems of that paper with the exception of (3), which actually strengthens Theorem 1.1.

(1) p. 379, last line: should read "formal adjoint of d"

(2) p. 381, lines 29, 30: should read: "unless ω is a 1-form". Similarly, p. 382, line above Eq. (1.6): should read "$p = 1$".

(3) p. 381, last paragraph: Add "The constant m^2 can be taken to be zero except in Eqs. (1.7) and (1.8). In (1.7), take Q as defined here; otherwise take $Q = |F_A|^2 \geq 0$".

(4) p. 382, two lines below Eq. (1.6): should read "$|w|^2 < 2/(\gamma + 1)$".

(5) p. 387, equation following inequality (3.5): should read

$$-L(Q) = \frac{\partial}{\partial x^i}\left(a^{ij}\frac{\partial Q}{\partial x^j}\right).$$

(6) p. 387, inequality (3.6): should read

$$\int_B \sum_{i,j} a^{ij}\frac{\partial Q}{\partial x^i}\frac{\partial \xi}{\partial x^j} - C_1(|A| + |\theta|)\left(\sum_j \left|\frac{\partial Q}{\partial x^j}\right|\right)\zeta$$
$$-C_2(|\nabla A| + |F_A|)Q\zeta d^n x \leq 0.$$

The statements leading to this inequality, beginning with Eq. (3.1), should be replaced by the proof of Proposition 4.1 of the present article (in which θ is defined). The corrected inequality (3.6) should be carried through the calculations of Lemma 3.2, but the new terms added here are treated in exactly the same way as the original terms, so this correction does not affect the subsequent analysis in any essential way. All the assertions in the proof about inequality (3.6) apply to the corrected version.

(7) p. 394, heading for Section 4: should read "The Hölder continuity of the curvature".

References

[1] W. Allard, On the first variation of a varifold, Ann. Math. 95 (1972) 417–491.
[2] D. Costa, G. Liao, On the removability of a singular submanifold for weakly harmonic maps, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. 35 (1988) 321–344.
[3] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (1981) 525–598.
[4] M. Henneaux, C. Teitelboim, p-form electrodynamics, Found. Phys. 16 (7) (1986) 593–617.
[5] C. Johnson, Numerical Solutions of Partial Differential Equations by the Finite Element Method, Cambridge University Press, Cambridge, 1987.
[6] A. Lichnerowicz, Courbure, nombres de Betti espaces symmetriques, in: Proceedings of the International Congress of Mathematicians, vol. 2, 1952, pp. 216–223.
[7] K.B. Marathe, G. Martucci, The Mathematical Foundations of Gauge Theories, North-Holland, Amsterdam, 1992.
[8] A. Milani, R. Picard, Decomposition theorems and their application to non-linear electro- and magneto-static boundary value problems, in: S. Hildebrandt, R. Leis (Eds.), Partial Differential Equations and Calculus of Variations, Lecture Notes in Mathematics, vol. 1357, Springer, Berlin, 1988.
[9] C.B. Morrey, Multiple Integrals in the Calculus of Variations, Springer, Berlin, 1966.
[10] T.H. Otway, An asymptotic condition for variational points of nonquadratic functionals, Ann. Fac. Sci. Toulouse Math. 11 (2) (1990) 187–195.
[11] T.H. Otway, Yang–Mills fields with nonquadratic energy, J. Geom. Phys. 19 (1996) 379–398.
[12] P. Price, A monotonicity formula for Yang–Mills fields, Manuscripta Math. 43 (1983) 417–491.
[13] J. Serrin, Local behavior of solutions of quasilinear equations, Acta Math. 111 (1964) 247–302.
[14] L.M. Sibner, An existence theorem for a nonregular variational problem, Manuscripta Math. 43 (1983) 45–72.

[15] L.M. Sibner, The isolated point singularity problem for the coupled Yang–Mills equations in higher dimensions, Math. Ann. 271 (1985) 125–131.
[16] L.M. Sibner, R.J. Sibner, A nonlinear Hodge–de Rham theorem, Acta Math. 125 (1970) 50–73.
[17] L.M. Sibner, R.J. Sibner, A maximum principle for compressible flow on a surface, Proc. Amer. Math. Soc. 71 (1) (1978) 103–108.
[18] L.M. Sibner, R.J. Sibner, Nonlinear Hodge theory: Applications, Adv. in Math. 31 (1979) 1–15.
[19] L.M. Sibner, R.J. Sibner, A sub-elliptic estimate for a class of invariantly defined elliptic systems, Pacific J. Math. 94 (2) (1981) 417–421.
[20] L.M. Sibner, R.J. Sibner, Classification of singular Sobolev connections by their holonomy, Comm. Math. Phys. (1992).
[21] K. Uhlenbeck, Connections with L^p bounds on curvature, Comm. Math. Phys. 83 (1982) 31–42.
[22] H. Wallin, A connection between α-capacity and L^p-classes of differentiable functions, Ark. Mat. 5 (24) (1964) 331–341.